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Mass Loss by Gravitational Radiation in Synge's Theory

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Using Synge's definition of mass for material systems, the variation of mass due to the radiation of gravitational waves is derived to any order of approximation with respect to the parameter associated with the weakness of the gravitational field generated. When certain additional boundary conditions are imposed, the result is analogous to the quadrupole formula and reduces to it in the first approximation.

1. INTRODUCTION

Approximation methods have been basically used for the analysis of mass loss due to the radiation of gravitational waves. Thus, using the linearized theory of general relativity, Einstein (1918) derived the celebrated quadrupole formula (which, therefore, is applicable only to nongravitationally bounded systems) and after the work of Einstein *et al.* (1938) the method of successive approximations was applied to describe the motion of bounded systems until Goldberg (1955) showed that this method was inadequate for the study of gravitational radiation.

In spite of this, the validity of the quadrupole formula has been confirmed in several works, such as those of Landau and Lifshitz (1951), Trautmann (1958), Peres (1950*a,b*, 1960), Peters (1964), Burke (1969), Thorne (1969*a,b*), and Chandrasekhar and Esposito (1970), while the works of Hu (1947), Sheidegger (1953), and Smith and Havas (1965), in which the results contradict the validity of this formula, have been shown to be incomplete.

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Nevertheless, the work of Chandrasekhar and Esposito (1970) has been criticized by Ehlers *et al.* (1976) as the field equations have not been used systematically. The method of Chandrasekhar and Esposito has the inconvenience of the appearance of divergent integrals just in the order of approximation where energy loss has to be taken into account, and although this criticism still has not led to a mathematically rigorous derivation of the formula, it has led to improved derivations (Walker and Will, 1980; Anderson, 1980; Damour, 1983; Cooperstock and Lim, 1985; Futamase and Schutz, 1985; Winicour, 1987).

The inconvenience of divergent integrals has been avoided (up to and including the $2\frac{1}{2}$ post-Newtonian approximation) by using a modification of Anderson and Decanio's (1975) method proposed by Ehlers (1978) and developed by Kerlich (1980*a,b*).

By this method, which is analogous to the "Hilbert type" expansion used to obtain the so-called asymptotic series solution of the Boltzmann integrodifferential equation, Brueuer and Rudolf (1981) have evaluated the energy loss for a radiating system from the radiation damping force, verifying the validity of Einstein's classic formula for quasiperiodic motions. Papapetrou and Linet (1981) have also obtained the same result in an independent way by using the so-called Papapetrou (1951) field equations generalizing those of the method of Fock (1939).

All these methods, besides having the common characteristic of using series expansions for the metric in weak fields, use from the beginning the slow motion condition. In Synge's method, which is a variant of the "fast approximation," series expansions of the metric are abandoned. Furthermore, by suitable initial conditions the appearance of divergent integrals is avoided to any order of approximation (Synge, 1970).

With the slow motion condition, McCrea (1981) has applied this method to evaluate the energy loss by gravitational radiation and has obtained this loss from the radiation damping force in the $2\frac{1}{2}$ post-Newtonian approximation.

The present state of the general problem, and in particular the questions concerning the use of series expansions, the appearance of divergent integrals, and the slow motion condition can be found in Persides (1987*a,b*).

Here, following the line of reasoning in Gambi *et al.* (1987), by using Synge's method without the slow motion condition, and maintaining an arbitrary order of approximation, the energy flux through a sphere of radius sufficiently large (in a technical sense which will be prescribed) is derived. The result is analogous to the familiar quadrupole formula and reduces to it in the first approximation.

The method is briefly summarized in Section 2. In Section 3 the asymptotic expression for the so-called truncated Einstein pseudotensor is evaluated up to the $K^{N+1}r^{-2}$ order of approximation. The variation of mass due to the radiation of gravitational waves is derived in Section 4. In Section 5 the result is compared with the classical one. The asymptotic expression for the truncated Einstein pseudotensor is detailed in an Appendix.

2. DESCRIPTION OF THE MODEL AND NOTATION

We consider a 4-space with coordinates x_a and use imaginary time $x_4 = it$ so that the formal signature of the metric δ_{ab} is +4. Latin indices take the values 1, 2, 3 and 4, and Greek indices the values 1, 2 and 3, with the summation convention in each case. Partial derivatives with respect to coordinates are indicated by commas, and units are chosen so that both the gravitational constant and the speed of light equal one.

In Synge's method (Synge, 1970) the space-time metric is considered to be of the form

$$g_{ab} = \delta_{ab} + \gamma_{ab} \tag{1}$$

so that the truncated Einstein pseudotensor $\hat{G}^{ab}$ is defined by

$$G^{ab} := L_{ab} + \hat{G}^{ab} \tag{2}$$

where L_{ab} is the linear part of the Einstein tensor, i.e.,

$$L_{ab} := \tfrac{1}{2}(\gamma_{ab,cc} + \gamma_{cc,ab} - \gamma_{ac,cb} - \gamma_{bc,ca}) - \tfrac{1}{2}\delta_{ab}(\gamma_{cc,dd} - \gamma_{cd,dc}) \tag{3}$$

so that it is always true that

$$L_{ab,b} = 0 \tag{4}$$

If the coordinate conditions

$$\gamma^*_{ab,b} = 0 \tag{5}$$

are imposed, with

$$\gamma^*_{ab} := \gamma_{ab} - \tfrac{1}{2}\delta_{ab}\gamma_{cc} \tag{6}$$

we have

$$L_{ab} = \tfrac{1}{2}\Box\gamma^*_{ab} \tag{7}$$

Here the symbol $\Box := \nabla^2 + \partial_{44}$ is the wave operator in the flat space-time. The gravitational field equations become

$$\Box\gamma^*_{ab} = -2\kappa H^{ab} \qquad (\kappa = 8\pi) \tag{8}$$

where H^{ab} is the energy-momentum complex defined by

$$H^{ab} := T^{ab} + \kappa^{-1}\hat{G}^{ab} \tag{9}$$

T^{ab} is the energy-momentum tensor for the material system generating the field. Equations (9) constitute the basis of Synge's approximation.

The retarded integral operator J is defined by

$$Jf(\mathbf{x}, t) := -\frac{1}{4\pi} \int f(\mathbf{x}', t') \|\mathbf{x}-\mathbf{x}'\|^{-1} \, d_3x' \tag{10}$$

$$(t' := t - \|\mathbf{x}-\mathbf{x}'\|)$$

where f is such that both differentiation and integration are valid, so that

$$\Box Jf = J \Box f = f \tag{11}$$

$\mathbf{x}$ represents a 3-vector in the flat space-time and $\|\mathbf{x}\|$ its Euclidean norm.

Now, considering the sequence of metrics

$$\underset{M}{g}_{ab} = \delta_{an} + \underset{M}{\gamma}_{ab} \qquad (M = 0, \ldots, N) \tag{12}$$

and the sequence of energy complexes

$$\underset{M}{H}^{ab} = T^{ab} + \kappa^{-1} \underset{M}{\hat{G}}^{ab} \qquad (M = 1, \ldots, N) \tag{13}$$

(M means that the quantity it accompanies is calculated with the metric tensor $\underset{M}{g}_{ab}$), then, if the energy-momentum tensor is chosen so that the equations of motion are satisfied in the Nth approximation, i.e.,

$$\underset{N-1}{H}^{ab}{}_{,b} = 0 \tag{14}$$

we obtain

$$\underset{N}{\gamma}^{*}_{ab} = -2\kappa J \underset{N-1}{H}^{ab} \tag{15}$$

$\underset{M}{\gamma}_{ab}$ satisfies the coordinate conditions

$$\underset{M}{\gamma}^{*}_{ab,b} = 0 \tag{16}$$

The weak field approximation is introduced by assuming

$$T^{ab} = O(k) \tag{17}$$

where k is a small constant in the sense explained by Synge (1970). Then, it can be demonstrated that

$$\underset{N-1}{T^{ab}} \Big|_{b} = 0$$

(where $|_{N-1}$ means covariant derivative calculated with the metric $\underset{N-1}{g_{ab}}$) implies

$$\underset{N-1}{H^{ab}}{}_{,b} = O(k^{N+1}) \tag{18}$$

and, furthermore, condition (17) establishes that

$$\underset{M}{\hat{G}^{ab}} - \underset{M-1}{\hat{G}^{ab}} = O(k^{M+1}) \qquad (M = 1, \ldots, N) \tag{19}$$

If the total mass of the material system is defined by

$$m := -\int_{x_4=\mathrm{Ct.}} \underset{N-1}{H^{44}}\, d_3x \tag{20}$$

equation (18) let us write the variation of total mass in the form

$$\frac{dm}{dx_4} = \oint (T^{4\alpha} + \kappa^{-1}\hat{G}^{\alpha 4})\, d\Sigma_\alpha + O(k^{N+1}) \tag{21}$$

where the integral on the right-hand side is taken over a sphere of radius large enough around the material system, and whose surface element has been written as $d\Sigma_\alpha$. If both the conditions that the material system is of completely mechanical origin and that the matter is localized in a bounded region of space are imposed, then T^{ab} is of compact support, so that choosing the surface of integration far away from the material system, we obtain from (21)

$$\frac{dm}{dx_4} = \kappa^{-1} \oint \underset{N-1}{\hat{G}^{\alpha 4}}\, d\Sigma_\alpha + O(k^{N+1}) \tag{22}$$

for, due to (19), in order to obtain the variation of mass up to $O(k^{N+1})$, $\underset{N}{\hat{G}^{ab}}$ or $\underset{N-1}{\hat{G}^{ab}}$ can be used indistinguishably.

3. ASYMPTOTIC EXPRESSIONS FOR THE TRUNCATED EINSTEIN PSEUDOTENSOR

3.1. Metric Deviations Far from the Material System

Let $r := \|\mathbf{x}\|$ be the radius of a sphere large enough so that $\|\mathbf{x}\| \gg \|\mathbf{x}'\|$. Then

$$\|\mathbf{x} - \mathbf{x}'\|^{-1} = \|\mathbf{x}\|^{-1} + O(r^{-2}) \tag{23}$$

and the Nth term of the metric sequence (15) becomes

$$\underset{N}{\gamma^{*}_{ab}} = \frac{\kappa}{2\pi r} \int \underset{N-1}{H^{ab}}(\mathbf{x}', t')\, d_3x' + O(r^{-2}) \tag{24}$$

from which we have

$$\underset{N}{\gamma}{}^{*}_{ab,c}=\frac{\kappa}{2\pi r}\left[\int \frac{\partial}{\partial t'}\underset{N-1}{H^{ab}}\,d_3x'\right]\frac{\partial t'}{\partial x_c}+O(r^{-2}) \tag{25}$$

Using now the fact that

$$\frac{\partial t'}{\partial x_4}=-i, \qquad \frac{\partial t'}{\partial x_\alpha}=-\frac{\partial\|\mathbf{x}-\mathbf{x}'\|}{\partial x_\alpha}=-\frac{x_\alpha}{r}+O(r^{-2}) \tag{26}$$

from (25) we have

$$\underset{N}{\gamma}{}^{*}_{ab,\alpha}=-\frac{\kappa}{2\pi r}n_\alpha\int \frac{\partial}{\partial t'}\underset{N-1}{H^{ab}}\,d_3x'+O(r^{-2}) \tag{27}$$

and

$$\underset{N}{\gamma}{}^{*}_{ab,4}=-\frac{\kappa}{2\pi r}i\int \frac{\partial}{\partial t'}\underset{N-1}{H^{ab}}\,d_3x'+O(r^{-2}) \tag{28}$$

where n_α is the unit radial vector

$$n_\alpha := x_\alpha/r \tag{29}$$

From the comparison between (27) and (28) we obtain the known asymptotic relationship between the spatial and time derivatives,

$$\underset{N}{\gamma}{}^{*}_{ab,\alpha}=-in_\alpha\underset{N}{\gamma}{}^{*}_{ab,4} \tag{30}$$

Furthermore, it has been verified that the potentials $\underset{N}{\gamma}{}^{*}_{ab}$ and their derivatives are proportional to r^{-1} asymptotically. This fact, together with equation (7), leads us to see that, asymptotically,

$$\underset{N}{\gamma}{}^{*}_{ab,cc}=O(r^{-2}) \tag{31}$$

whatever the value of N.

3.2. Asymptotic Expression for $\underset{2}{\hat{G}}{}^{ab}$

In order to obtain the asymptotic expression for the truncated pseudotensor $\underset{2}{\hat{G}}{}^{ab}$, we use the metric for the first approximation

$$\underset{1}{g}_{ab}=\delta_{ab}+\underset{1}{\gamma}_{ab} \tag{32}$$

This metric satisfies the coordinate conditions

$$\underset{1}{\gamma}{}^{*}_{ab,b}=O(k^2) \tag{33}$$

which are equivalent to

$$[bb, a]_1 = O(k^2) \tag{34}$$

where $[bb, a]_1$ are the Christoffel symbols of the first kind for the metric (32). In this order of approximation, $\underset{2}{\hat{G}}^{ab}$ is given by (Synge, 1969)

$$\underset{2}{\hat{G}}^{ab} = M_{ab} - \underset{1}{\gamma}_{rs} L_{rabs} + \tfrac{1}{2} \underset{1}{\gamma}_{ab} L^*_{rr} + \delta_{ab} \underset{1}{\gamma}_{rs} L^*_{rs} - (\underset{1}{\gamma}_{ar} L^*_{rb} + \underset{1}{\gamma}_{br} L^*_{ra}) + O(k^3) \tag{35}$$

where the asterisks stand for the conjugate tensor of a symmetric tensor. Since in general the conjugate A^*_{ab} for a tensor A_{ab} is defined by

$$A^*_{ab} := A_{ab} - \tfrac{1}{2} \delta_{ab} A_{cc} \tag{36}$$

then, in the order of approximation considered, the conjugate of the first term in the right-hand side of (34) is given by

$$M^*_{ab} := -[am, m]_1 [bm, m]_1 + O(k^3) \tag{37}$$

while the linear part L_{abcd} of the Riemann tensor is given, in any order of approximation, by

$$L_{abcd} := \tfrac{1}{2} (\gamma_{ad,bc} + \gamma_{bc,ad} - \gamma_{ac,bd} - \gamma_{bd,ac}) \tag{38}$$

Now, since between (3) and (38) there exists the relationship

$$L_{bc} = L_{pbcp} - \tfrac{1}{2} \delta_{bc} L_{pqqp} \tag{39}$$

then, using (6) and (7), we can write nonlinear part (35) for the Einstein tensor in the form

$$\begin{aligned}
\underset{2}{\hat{G}}^{ab} = {} & \underset{*}{M}_{ab} - (\underset{1}{\gamma}^*_{rs} - \tfrac{1}{2} \delta_{rs} \underset{1}{\gamma}^*_{cc}) \underset{*}{L}_{rabs} \\
& + \tfrac{1}{4} (\underset{1}{\gamma}^*_{ab} - \tfrac{1}{2} \delta_{ab} \underset{1}{\gamma}^*_{cc}) \Box (\underset{1}{\gamma}^*_{rr} - \tfrac{1}{2} \delta_{rr} \underset{1}{\gamma}^*_{cc}) \\
& + \tfrac{1}{2} \delta_{ab} (\underset{1}{\gamma}^*_{rs} - \tfrac{1}{2} \delta_{rs} \underset{1}{\gamma}^*_{cc}) \Box (\underset{1}{\gamma}^*_{rs} - \tfrac{1}{2} \delta_{rs} \underset{1}{\gamma}^*_{cc}) \\
& - \tfrac{1}{2} [(\underset{1}{\gamma}^*_{ar} - \tfrac{1}{2} \delta_{ar} \underset{1}{\gamma}^*_{cc}) \Box (\underset{1}{\gamma}^*_{rb} - \tfrac{1}{2} \delta_{rb} \underset{1}{\gamma}^*_{cc}) \\
& + (\underset{1}{\gamma}^*_{br} - \tfrac{1}{2} \delta_{br} \underset{1}{\gamma}^*_{cc}) \Box (\underset{1}{\gamma}^*_{ar} - \tfrac{1}{2} \delta_{ar} \underset{1}{\gamma}^*_{cc})] + O(k^3)
\end{aligned} \tag{40}$$

The asterisks below M_{ab} and L_{rabs} mean that in definitions (36) and (37), instead of the metric deviations γ_{ab}, their duals are used. Now, taking into account (31), we obtain from (40) that, in the first approximation with respect to the parameter k, the truncated pseudotensor has the following asymptotic expression with respect to distance:

$$^{(\infty)}\underset{2}{\hat{G}}{}^{ab} = \underset{*}{M}_{ab} - (\underset{1}{\gamma}{}^{*}_{rs} - \tfrac{1}{2}\delta_{rs}\underset{1}{\gamma}{}^{*}_{cc})\,\underset{*}{L}_{rabs} + O(k^3 r^{-2}) \tag{41}$$

3.3. Asymptotic Expression for $\underset{N}{\hat{G}}{}^{ab}$

We now generalize (41) to the case where the generic metric (12) is considered, M being an arbitrary natural number. Since for the evaluation of the variation of mass given in (22), the sphere surrounding the system is of arbitrary radius, we choose this radius so that formulas (24) and (25) may be used, maintaining all the terms in $\underset{N}{\hat{G}}{}^{ab}$ greater than $k^{N+1}r^{-2}$ and satisfying that the ones of order k^2r^{-3} are negligible. These conditions, which are satisfied if $r > k^{1-N}$, can always be assumed in Synge's method because, since it is of the fast approximation type, it leads to solutions for the Einstein equations valid all over the space. So, any term of order k^Pr^{-Q}, with $3 \leq P \leq N$ and $Q \geq 2$, appearing in the asymptotic expression of $-\underset{N-1}{G}{}^{ab}$ will be negligible.

From now on, we assume coordinate conditions of the form

$$\underset{N}{\gamma}{}^{*}_{ab,b} = O(k^{N+1}) \qquad (\text{pr} \quad [bb, a] = O(k^{N+1})) \tag{42}$$

assuring in this way the satisfaction of the following condition:

$$L_{ab} = \tfrac{1}{2}\Box\,\underset{N}{\gamma}{}^{*}_{ab} + O(k^{N+1}) \tag{43}$$

which is analogous to (7). Then, using (43) in the expression for the truncated Einstein pseudotensor (see Appendix),

$$^{(\infty)}\underset{N}{\hat{G}}{}^{ab} = -\underset{N}{\gamma}_{ij}L_{iabj} - \underset{N}{\gamma}_{bc}L^{*}_{ac} - \underset{N}{\gamma}_{ac}L^{*}_{bc}$$

$$+\,\delta_{ab}\underset{N}{\gamma}_{rs}L^{*}_{rs} + \tfrac{1}{2}\underset{N}{\gamma}_{ab}L^{*}_{rr} + M_{ab} + O(k^{N+1}r^{-2}) \tag{14}$$

and following a reasoning similar to the one that led to (41), we obtain

$$^{(\infty)}\underset{N}{\hat{G}}{}^{ab} = \underset{*}{M}_{ab} - \underset{N}{\gamma}{}^{*}_{rs}\underset{*}{L}_{rabs} + \tfrac{1}{2}\underset{N}{\gamma}{}^{*}_{cc}\underset{*}{L}_{rabs} + O(k^{N+1}r^{-2}) \tag{45}$$

so that it is clear that, in order to obtain the value of (45) as a function of the metric deviations, we need to calculate the corresponding expressions for $\underset{*}{M}_{ab}$ and $\underset{*}{L}_{abcd}$.

Since M_{ab} is defined by

$$M_{ab} := M_{mabm} - \tfrac{1}{2}\delta_{ab}M_{mrrm} \tag{46}$$

with

$$M_{abcd} := [ad, m][bc, m] - [ac, m][ba, m] \tag{47}$$

then, taking into account (42), we have

$$M_{ab} = [mb, l][am, l] + \tfrac{1}{2}\delta_{ab}[mr, l][rm, l] + O(k^{N+1}) \tag{48}$$

so that, substituting the metric in (12) with $M = N$ in (48), we have

$$\begin{aligned} M_{ab} = &-\tfrac{1}{4}[\underset{N}{\gamma}_{ml,b}\underset{N}{\gamma}_{ml,a} + 2\underset{N}{\gamma}_{bl,m}\underset{N}{\gamma}_{al,m} - 2\underset{N}{\gamma}_{mb,l}\underset{N}{\gamma}_{al,m}] \\ &+\tfrac{1}{8}\delta_{ab}[3\underset{N}{\gamma}_{ml,r}\underset{N}{\gamma}_{ml,r} - 2\underset{N}{\gamma}_{mr,l}\underset{N}{\gamma}_{rl,m}] + O(k^{N+1}r^{-2}) \end{aligned} \tag{49}$$

and from here we have

$$\begin{aligned} \underset{*}{M}_{ab} = &-\tfrac{1}{4}\underset{N}{\gamma}^{*}_{ml,b}\,\underset{N}{\gamma}^{*}_{ml,a} + \tfrac{1}{8}\underset{N}{\gamma}^{*}_{cc,a}\,\underset{N}{\gamma}^{*}_{dd,b} - \tfrac{1}{2}\underset{N}{\gamma}^{*}_{bl,m}\,\underset{N}{\gamma}^{*}_{al,m} \\ &+\tfrac{1}{2}\underset{N}{\gamma}^{*}_{ab,m}\,\underset{N}{\gamma}^{*}_{cc,m} - \tfrac{1}{4}\underset{N}{\gamma}^{*}_{al,b}\,\underset{N}{\gamma}^{*}_{cc,l} + \tfrac{1}{2}\underset{N}{\gamma}^{*}_{mb,l}\,\underset{N}{\gamma}^{*}_{al,m} \\ &-\tfrac{1}{4}\underset{N}{\gamma}^{*}_{mb,a}\,\underset{N}{\gamma}^{*}_{dd,m} + \tfrac{3}{16}\underset{N}{\gamma}^{*}_{cc,m}\,\underset{N}{\gamma}^{*}_{dd,m}\delta_{ab} \\ &+\tfrac{3}{8}\underset{N}{\gamma}^{*}_{ml,r}\,\underset{N}{\gamma}^{*}_{ml,r}\delta_{ab} - \tfrac{1}{4}\underset{N}{\gamma}^{*}_{mr,l}\,\underset{N}{\gamma}^{*}_{rl,m}\delta_{ab} \\ &+\tfrac{1}{4}\underset{N}{\gamma}^{*}_{rl,r}\,\underset{N}{\gamma}^{*}_{cc,l}\delta_{ab} + O(k^{N+1}r^{-2}) \end{aligned} \tag{50}$$

The expression for $\underset{*}{L}_{rabs}$ is obtained from the linear part of the Riemann tensor defined in (38) by substituting the γ's by the γ^{*}'s. The result is

$$\begin{aligned} \underset{*}{L}_{rabs} = &\tfrac{1}{2}(\underset{N}{\gamma}^{*}_{rs,ab} + \underset{N}{\gamma}^{*}_{ab,rs} - \underset{N}{\gamma}^{*}_{rb,as} - \underset{N}{\gamma}^{*}_{as,rb}) \\ &-\tfrac{1}{4}(\delta_{rs}\underset{N}{\gamma}^{*}_{cc,ab} + \delta_{ab}\underset{N}{\gamma}^{*}_{cc,rs} - \delta_{rb}\underset{N}{\gamma}^{*}_{cc,as} - \delta_{as}\underset{N}{\gamma}^{*}_{cc,rb}) + O(k^{N+1}r^{-2}) \end{aligned} \tag{51}$$

Then, from (31), (44), (49) and (50) we finally have, as asymptotic form for the truncated Einstein pseudotensor of order N, the following value:

$$
\begin{aligned}
{}^{(\infty)}\underset{N}{\hat{G}}{}^{ab} = & -\tfrac{1}{4}\underset{N}{\gamma}{}^{*}_{ml,b}\,\underset{N}{\gamma}{}^{*}_{ml,a} + \tfrac{1}{8}\underset{N}{\gamma}{}^{*}_{cc,a}\,\underset{N}{\gamma}{}^{*}_{dd,b} - \tfrac{1}{2}\underset{N}{\gamma}{}^{*}_{bl,m}\,\underset{N}{\gamma}{}^{*}_{al,m} \\
& + \tfrac{1}{2}\underset{N}{\gamma}{}^{*}_{ab,m}\,\underset{N}{\gamma}{}^{*}_{cc,m} - \tfrac{1}{4}\underset{N}{\gamma}{}^{*}_{al,b}\,\underset{N}{\gamma}{}^{*}_{cc,l} + \tfrac{1}{2}\underset{N}{\gamma}{}^{*}_{mb,l}\,\underset{N}{\gamma}{}^{*}_{al,m} \\
& - \tfrac{1}{4}\underset{N}{\gamma}{}^{*}_{mb,a}\,\underset{N}{\gamma}{}^{*}_{dd,m} + \tfrac{3}{16}\underset{N}{\gamma}{}^{*}_{cc,m}\,\underset{N}{\gamma}{}^{*}_{dd,m}\delta_{ab} \\
& + \tfrac{3}{8}\underset{N}{\gamma}{}^{*}_{ml,r}\,\underset{N}{\gamma}{}^{*}_{ml,r}\delta_{ab} - \tfrac{1}{4}\underset{N}{\gamma}{}^{*}_{mr,l}\,\underset{N}{\gamma}{}^{*}_{rl,m}\delta_{ab} \\
& + \tfrac{1}{4}\underset{N}{\gamma}{}^{*}_{rl,r}\,\underset{N}{\gamma}{}^{*}_{cc,l}\delta_{ab} - \tfrac{1}{2}\underset{N}{\gamma}{}^{*}_{rs}\,\underset{N}{\gamma}{}^{*}_{rs,ab} - \tfrac{1}{2}\underset{N}{\gamma}{}^{*}_{rs}\,\underset{N}{\gamma}{}^{*}_{ab,rs} \\
& + \tfrac{1}{2}\underset{N}{\gamma}{}^{*}_{rs}\,\underset{N}{\gamma}{}^{*}_{rb,as} + \tfrac{1}{2}\underset{N}{\gamma}{}^{*}_{rs}\,\gamma^{*}_{as,rb} + \tfrac{1}{4}\underset{N}{\gamma}{}^{*}_{rr}\,\underset{N}{\gamma}{}^{*}_{cc,ab} \\
& + \tfrac{1}{4}\underset{N}{\gamma}{}^{*}_{rs}\,\underset{N}{\gamma}{}^{*}_{cc,rs}\delta_{ab} - \tfrac{1}{4}\underset{N}{\gamma}{}^{*}_{bs}\,\underset{N}{\gamma}{}^{*}_{cc,ab} \\
& - \tfrac{1}{4}\underset{N}{\gamma}{}^{*}_{ra}\,\underset{N}{\gamma}{}^{*}_{cc,rb} + O(k^{N+1}r^{-2})
\end{aligned}
\tag{52}
$$

It is to be noted that, unlike in the linearized theory of Landau and Lifshitz (1951), both the first and second derivatives of the metric deviations appear in (52).

4. MASS LOSS DUE TO THE GRAVITATIONAL RADIATION

In this section we use (52) to study mass loss due to the radiation of gravitational waves. To this end, the mean time value for the mass variation of a material system having an approximately periodic motion of period τ is evaluated. In general τ is given by

$$\left\langle \frac{dm}{dt} \right\rangle_{\tau} = \int_{\tau} \frac{dm}{dt}\,dt \tag{53}$$

If equation (22) is taken into account, the integrand in (53) can be written in the form

$$\frac{dm}{dt} = i\kappa^{-1} \int^{(\infty)} \underset{N}{\hat{G}}{}^{\alpha 4}\, d\Sigma_{\alpha} \tag{54}$$

so that from (53) we have

$$\left\langle\frac{dm}{dt}\right\rangle_\tau = \kappa^{-1}\int d\Sigma_\alpha\left(\int_\tau^{(\infty)} \underset{N}{\hat{G}}{}^{\alpha r} i\,dt\right) \equiv \kappa^{-1}\int d\Sigma_\alpha\left(\int^{(\infty)} \underset{N}{\hat{G}}{}^{\alpha 4}\,dx_4\right) \quad (55)$$

Then, taking into account (52), we have

$$\begin{aligned}\left\langle\frac{dm}{dt}\right\rangle_\tau = \kappa^{-1}\int d\Sigma_\alpha \int [&-\tfrac{1}{4}\underset{N}{\gamma}{}^*_{ml,4}\underset{N}{\gamma}{}^*_{ml,\alpha} + \tfrac{1}{8}\underset{N}{\gamma}{}^*_{cc,\alpha}\underset{N}{\gamma}{}^*_{dd,4}\\ &-\tfrac{1}{2}\underset{N}{\gamma}{}^*_{4l,m}\underset{N}{\gamma}{}^*_{\alpha l,m} + \tfrac{1}{2}\underset{N}{\gamma}{}^*_{\alpha 4,m}\underset{N}{\gamma}{}^*_{cc,m} - \tfrac{1}{4}\underset{N}{\gamma}{}^*_{\alpha l,4}\underset{N}{\gamma}{}^*_{cc,l}\\ &+\tfrac{1}{2}\underset{N}{\gamma}{}^*_{m4,l}\underset{N}{\gamma}{}^*_{\alpha l,m} - \tfrac{1}{4}\underset{N}{\gamma}{}^*_{m4,\alpha}\underset{N}{\gamma}{}^*_{dd,m} - \tfrac{1}{2}\gamma^*_{rs}\underset{N}{\gamma}{}^*_{rs,\alpha 4}\\ &-\tfrac{1}{2}\underset{N}{\gamma}{}^*_{rs}\underset{N}{\gamma}{}^*_{\alpha 4,rs} + \tfrac{1}{2}\underset{N}{\gamma}{}^*_{rs}\underset{N}{\gamma}{}^*_{r4,\zeta s} + \tfrac{1}{2}\underset{N}{\gamma}{}^*_{rs}\gamma^*_{\alpha s,r4}\\ &+\tfrac{1}{4}\underset{N}{\gamma}{}^*_{rr}\underset{N}{\gamma}{}^*_{cc,\alpha 4} - \tfrac{1}{4}\underset{N}{\gamma}{}^*_{4s}\underset{N}{\gamma}{}^*_{cc,\alpha s} - \tfrac{1}{4}\underset{N}{\gamma}{}^*_{r\alpha}\underset{N}{\gamma}{}^*_{cc,r4}]\,dx_4\\ &+O(k^{N+1}r^{-2})\end{aligned} \quad (56)$$

Now, since equation (30) can be written in the form

$$\gamma^*_{ab,r} = N_r\gamma^*_{ab,4} \quad (57)$$

where N_r is the null 4-vector given by

$$N_r := (-in_\alpha, 1) \quad (58)$$

and applying the method of integration by parts together with (31), we have

$$\int_0^\tau \underset{N}{\gamma}{}^*_{pq,m}\underset{N}{\gamma}{}^*_{rs,m}\,dx_4 = \Delta(\underset{N}{\Gamma}{}_{rs}N_m\underset{N}{\gamma}{}^*_{pq,m}) + O(k^{N+1}r^{-2}) \quad (59)$$

In the same way, but now using the coordinate conditions (42), we have

$$\int_0^\tau \underset{N}{\gamma}{}^*_{pq,m}\underset{N}{\gamma}{}^*_{rs,q}\,dx_4 = \Delta(\underset{N}{\Gamma}{}_{rs}N_q\underset{N}{\gamma}{}^*_{pq,m}) + O(k^{N+1}r^{-2}) \quad (60)$$

and

$$\int_0^\tau \underset{N}{\gamma}{}^*_{ab}\underset{N}{\gamma}{}^*_{cd,ae}\,dx_4 = \Delta(\underset{N}{\Gamma}{}_{cde}N_a\underset{N}{\gamma}{}_{ab}) + O(k^{N+1}r^{-2}) \quad (61)$$

where $\underset{N}{\Gamma}{}_{rs}$ and $\underset{N}{\Gamma}{}_{cde}$ are given by

$$\underset{N}{\Gamma}{}_{rs} := \int \underset{N}{\gamma}{}^*_{rs,4}\,dx_4, \qquad \underset{N}{\Gamma}{}_{cde} := \int \gamma_{cd,e4}\,dx_4 \quad (62)$$

respectively. Then, using (59)-(61), (56) can be written in the form

$$\left\langle\frac{dm}{dt}\right\rangle_{\tau}=\kappa^{-1}\int d\Sigma_{\alpha}\left\{\int dx_4\left[-\tfrac{1}{4}\underset{N}{\gamma}^{*}_{ml,4}\underset{N}{\gamma}^{*}_{ml,4}+\tfrac{1}{8}\underset{N}{\gamma}^{*}_{cc,\alpha}\underset{N}{\gamma}^{*}_{dd,4}\right.\right.$$
$$\left.\left.+\tfrac{1}{2}\underset{N}{\Gamma}_{rs\alpha}\underset{N}{\gamma}^{*}_{rs,4}-\tfrac{1}{4}\underset{N}{\gamma}^{*}_{rr,4}\Gamma_{cc,\alpha 4}\right]+R(x^{\alpha},\tau)\right\} \tag{63}$$

where $R(x^{\alpha},\tau)$ is given by

$$R(x^{\alpha},\tau):=\Delta\left[-\tfrac{1}{2}\underset{N}{\gamma}^{*}_{4l,m}N_m\underset{N}{\Gamma}_{\alpha l}+\tfrac{1}{2}\underset{N}{\gamma}^{*}_{\alpha 4,m}N_m\underset{N}{\Gamma}_{cc}-\tfrac{1}{4}\underset{N}{\gamma}^{*}_{\alpha l,4}N_l\Gamma_{cc}\right.$$
$$+\tfrac{1}{2}\underset{N}{\gamma}^{*}_{\alpha l,m}N_l\underset{N}{\Gamma}_{m4}-\tfrac{1}{4}\underset{N}{\gamma}^{*}_{m4,\alpha}N_m\underset{N}{\Gamma}_{dd}-\tfrac{1}{2}\underset{N}{\gamma}^{*}_{rs}N_s\Gamma_{\alpha 4r}$$
$$+\tfrac{1}{2}\underset{N}{\gamma}^{*}_{rs}N_s\underset{N}{\Gamma}_{r4\alpha}+\tfrac{1}{2}\underset{N}{\gamma}^{*}_{rs}N_r\underset{N}{\Gamma}_{\alpha s4}-\tfrac{1}{4}\underset{N}{\gamma}^{*}_{4s}N_s\Gamma_{cc,\alpha}$$
$$\left.-\tfrac{1}{4}\gamma^{*}_{r\alpha}N_r\underset{N}{\Gamma}_{cc4}-\tfrac{1}{2}\underset{N}{\gamma}^{*}_{rs}\Gamma_{rs\alpha}+\tfrac{1}{4}\underset{N}{\gamma}^{*}_{rr}\Gamma_{cc\alpha}\right] \tag{64}$$

If $\underset{N}{\Gamma}_{rs}$ and $\underset{N}{\Gamma}_{cde}$ in (62) differ from $\underset{N}{\gamma}^{*}_{rs}$ and $\underset{N}{\gamma}^{*}_{cde}$, respectively, in quantities of the order $O(k^{N+1}r^{-2})$, and if $R(x^{\alpha},\tau)$ is of this same order, then from (63) we have

$$\left\langle\frac{dm}{dt}\right\rangle_{\tau}=\kappa^{-1}\int d\Sigma_{\alpha}\int\left[-\tfrac{1}{4}\underset{N}{\gamma}^{*}_{ml,4}\underset{N}{\gamma}^{*}_{ml,\alpha}+\tfrac{1}{8}\underset{N}{\gamma}^{*}_{cc,\alpha}\underset{N}{\gamma}^{*}_{dd,4}\right.$$
$$\left.-\tfrac{1}{2}\underset{N}{\gamma}^{*}_{rs}\underset{N}{\gamma}^{*}_{rs,\alpha 4}+\tfrac{1}{4}\underset{N}{\gamma}^{*}_{rr}\underset{N}{\gamma}^{*}_{cc,\alpha 4}\right]dx_4+O(k^{N+1}r^{-2}) \tag{65}$$

or, using again the integration by parts to the last two terms in the integral with respect to time,

$$\left\langle\frac{dm}{dt}\right\rangle_{\tau}=\kappa^{-1}\int d\Sigma_{\alpha}\int dx_4\left[-\tfrac{1}{4}\underset{N}{\gamma}^{*}_{ml,4}\underset{N}{\gamma}^{*}_{ml,\alpha}+\tfrac{1}{8}\underset{N}{\gamma}^{*}_{cc,\alpha}\underset{N}{\gamma}^{*}_{dd,4}\right.$$
$$\left.+\tfrac{1}{2}\underset{N}{\gamma}^{*}_{rs,4}\underset{N}{\gamma}^{*}_{rs,\alpha}-\tfrac{1}{4}\underset{N}{\gamma}^{*}_{rr,4}\underset{N}{\gamma}^{*}_{cc,\alpha}\right]+O(k^{N+1}r^{-2}) \tag{66}$$

that is,

$$\left\langle\frac{dm}{dt}\right\rangle_{\tau}=\frac{1}{4\pi}\int d\Sigma_{\alpha}\int dx_4\left[\left(\underset{N}{\gamma}^{*}_{\beta\varepsilon,4}\underset{N}{\gamma}^{*}_{\beta\varepsilon,\alpha}+2\underset{N}{\gamma}^{*}_{\beta 4,4}\underset{N}{\gamma}^{*}_{\beta 4,\alpha}+\underset{N}{\gamma}^{*}_{\beta 4,\alpha}+\underset{N}{\gamma}^{*}_{44,4}\underset{N}{\gamma}^{*}_{44,\alpha}\right)\right.$$
$$\left.-\tfrac{1}{2}\left(\underset{N}{\gamma}^{*}_{\beta\beta,\alpha}\underset{N}{\gamma}^{*}_{\varepsilon\varepsilon,4}+2\underset{N}{\gamma}^{*}_{\beta\beta,\alpha}\underset{N}{\gamma}^{*}_{44,4}+\underset{N}{\gamma}^{*}_{44,\alpha}\underset{N}{\gamma}^{*}_{44,4}\right)\right]+O(k^{N+1}r^{-2}) \tag{67}$$

or, by means of (42) and (30),

$$\left\langle\frac{dm}{dt}\right\rangle_\tau = -\frac{1}{4\pi}\int d\Sigma_\alpha \int dx_4\,[in_\alpha \underset{N}{\gamma}{}^*_{\beta\varepsilon,4}\underset{N}{\gamma}{}^*_{\beta\varepsilon,4}$$

$$-2in_\alpha \underset{N}{\gamma}{}^*_{\beta\varepsilon,4}\underset{N}{\gamma}{}^*_{\beta\sigma,4} + in_\alpha n_\beta n_\varepsilon n_\rho n_\sigma \underset{N}{\gamma}{}^*_{\beta\rho,4}\underset{N}{\gamma}{}^*_{\varepsilon\sigma,4}$$

$$-\frac{i}{2}n_\alpha \underset{N}{\gamma}{}^*_{\beta\beta,4}\underset{N}{\gamma}{}^*_{\varepsilon\varepsilon,4} + in_\alpha n_\varepsilon n_\sigma \underset{N}{\gamma}{}^*_{\beta\beta,4}\underset{N}{\gamma}{}^*_{\varepsilon\sigma,4}$$

$$-\frac{i}{2}n_\alpha n_\beta n_\varepsilon n_\rho n_\sigma \underset{N}{\gamma}{}^*_{\beta\rho,4}\underset{N}{\gamma}{}^*_{\varepsilon\sigma,4} + O(k^{N+1}r^{-2}) \tag{68}$$

Now, writing the surface element in the form $d\Sigma_\alpha = n_\alpha\, d\Sigma$ and taking into account that

$$\frac{1}{4\pi}\int n_\alpha\, d\Omega = 0, \qquad \frac{1}{4\pi}\int n_\alpha n_\beta\, d\Omega = \tfrac{1}{3}\delta_{\alpha\beta}$$

$$\frac{1}{4\pi}\int n_\alpha n_\beta n_\varepsilon\, d\Omega = 0 \tag{69}$$

$$\frac{1}{4\pi}\int n_\alpha n_\beta n_\varepsilon n_\delta\, d\Omega = \tfrac{1}{15}(\delta_{\alpha\beta}\delta_{\varepsilon\rho} + \delta_{\alpha\varepsilon}\delta_{\beta\rho} + \delta_{\alpha\rho}\delta_{\beta\varepsilon})$$

where $d\Omega$ is the corresponding element of solid angle, from (68) we have

$$\left\langle\left\langle\frac{dm}{dt}\right\rangle_\tau\right\rangle_\Omega = \tfrac{1}{60}\int dt\, r^2(3\underset{N}{\gamma}{}^*_{\beta\varepsilon,4}\underset{N}{\gamma}{}^*_{\beta\varepsilon,4} - \underset{N}{\gamma}{}^*_{\beta\beta,4}\underset{N}{\gamma}{}^*_{\sigma\sigma,4}) + O(k^{N+1}r^{-2}) \tag{70}$$

The metric deviations appearing in (70) can be evaluated as in Landau and Lifshitz (1951) or Misner *et al.* (1973). In effect, differentiating (18) with respect to x_4, we have

$$\underset{N-1}{H}{}^{44}{}_{,44} = \underset{N-1}{H}{}^{\alpha\beta}{}_{,\alpha\beta} + O(k^{N+1}) \tag{71}$$

so that

$$(\underset{N-1}{H}{}^{44}x_\alpha x_\beta)_{,44} = (\underset{N-1}{H}{}^{\alpha\beta}x_\sigma x_\tau)_{,\sigma\tau} - 2(\underset{N-1}{H}{}^{\alpha\sigma}x_\tau + \underset{N-1}{H}{}^{\alpha\tau}x_\sigma)_{,\alpha} + 2\underset{N-1}{H}{}^{\sigma\tau} + O(k^{N+1}) \tag{72}$$

Now, using the divergence theorem and that $\underset{N-1}{H}{}^{ab}$ is null at infinity, from (72) we have

$$\int \underset{N-1}{H}{}^{\sigma\tau}\, d_3x' = \frac{1}{2}\frac{d^2 I_{\sigma\tau}}{dx_4^2} + O(k^{N+1}) \tag{73}$$

where

$$I_{\sigma\tau} := \int \underset{N-1}{H^{44}} x_\sigma x_\tau \, d_3x' \tag{74}$$

are the quadrupole moments of order N for the material distribution. Then, by (24), we have

$$\underset{N}{\gamma^*_{\sigma\tau}} = \frac{2}{r}\frac{d^2 I_{\sigma\tau}}{dx_4^2} + O(k^{N+1}r^{-2}) \tag{75}$$

Substituting now (75) into (70), we have

$$\left\langle\left\langle\frac{dm}{dt}\right\rangle_\tau\right\rangle_\Omega = \frac{1}{5}\int dt \left[\left(\frac{d^3 I_{\sigma\tau}}{dx_4^3}\right)^2 - \frac{1}{3}\left(\frac{d^3 I_{\sigma\sigma}}{dx_4^3}\right)^2\right] + O(k^{N+1}r^{-2}) \tag{76}$$

and, if the free trace part for the quadrupole momentum

$$\mathcal{I}_{\sigma\tau} := \int \underset{N-1}{H^{44}} (x_\sigma x_\tau - \tfrac{1}{3}\delta_{\sigma\tau} x_\varepsilon x_\varepsilon) \, d_3x' \tag{77}$$

together with the identity

$$\left(\frac{d^3 \mathcal{I}_{\sigma\tau}}{dx_4^3}\right)^2 \equiv \left(\frac{d^3 I_{\sigma\tau}}{dx_4^3}\right)^2 - \frac{1}{3}\left(\frac{d^3 I_{\varepsilon\varepsilon}}{dx_4^3}\right)^2 \tag{78}$$

are used, from (76) we finally have

$$\left\langle\left\langle\frac{dm}{dt}\right\rangle_\tau\right\rangle_\Omega = \frac{1}{5}\int dt \left(\frac{d^2 \mathcal{I}_{\sigma\tau}}{dx_4^3}\right)^2 + O(k^{N+1}r^{-2}) \tag{79}$$

that is,

$$\left\langle\left\langle\frac{dm}{dt}\right\rangle_\tau\right\rangle_\Omega = -\frac{1}{5}\int (\dddot{\mathcal{I}}_{\sigma\tau})^2 \, dt + O(k^{N+1}r^{-2}) \tag{80}$$

the dots meaning, as usual, derivatives with respect to time.

5. CONCLUSION

We have obtained one expression for the mass loss due to the radiation of gravitational waves by deriving the energy flux through a sphere of radius greater than k^{1-N}. To this end, we have used Synge's method to determine both the asymptotic value for the metric and the truncated Einstein pseudotensor to the order N.

The value of energy loss due to the radiation of gravitational waves derived in (80) is formally identical to the quadrupole formula derived in the linear approximation. This formula corresponds in Synge's method to the first approximation. Nevertheless, the value in (80) for the moments has been calculated taking into account all the terms of order less than $k^{N+1}r^{-2}$. As can be seen in (74), these terms appear in the energy-momentum complex (13) when the orders of approximation M in (12) (with $M \geq 2$) are taken into account.

APPENDIX

In a differentiable manifold with metric g_{ab}, the Riemann tensor is given by

$$R_{abcd} := \tfrac{1}{2}(g_{ab,cd} + g_{bc,ad} - g_{ac,bd} - g_{bd,ac}) + g^{mn}([ad, m][bc, n] - [ac, m][bd, n]) \tag{A1}$$

or, equivalently, by

$$R_{abcd} := U_{abcd} + g^{mn} D_{abcdmn} \tag{A2}$$

where U represents the linear part in (A1), and D contains the terms composed by product of two or more components of the metric tensor. Using these symbols, we can write the Einstein tensor in the form

$$G^{ad} = g^{ab} g^{dc} g g^{ij} U_{ibcj} - \tfrac{1}{2} g^{ab} g^{dc} g_{bc} g^{rs} g_{kl} U_{krsl} + g^{ab} g^{dc} g^{ij} g^{mn} D_{ibcjmn} - \tfrac{1}{2} g^{ab} g^{dc} g_{bc} g^{rs} g^{kl} g^{mn} D_{krslmn} \tag{A3}$$

Substituting in (A3) the covariant and contravariant components of the metric tensor by

$$g_{ab} = \delta_{ab} + \underset{N}{\gamma}_{ab} \tag{A4}$$

and

$$g^{ab} = \delta_{ab} - \underset{N}{\gamma}_{ab} + g(\underset{N_1}{\gamma} \overset{M}{\cdots} \underset{N_M}{\gamma}) + O(k^{N+1}) \tag{A5}$$

respectively [where $g(\underset{N_1}{\gamma} \overset{M}{\cdots} \underset{N_M}{\gamma})$ is a sum whose terms are formed by products of two or more $\underset{N_i}{\gamma}$, $1 \leq N_i \leq N-1$] and separating in (A3) the two terms

$$\delta^{ab}\delta^{dc}\delta^{ij} U_{ibcj} - \tfrac{1}{2}\delta^{ab}\delta^{dc}\delta_{bc}\delta^{rs}\delta^{kl} U_{krsl} \tag{A6}$$

which give the linear part (3) of the Einstein tensor, we obtain as asymptotic expression for the truncated Einstein pseudotensor

$$\overset{(\infty)}{\underset{N}{\hat{G}}}{}^{ad} = -\underset{N}{\gamma}_{ij} U_{iadj} - \underset{N}{\gamma}_{dc} U_{iaci} - \underset{N}{\gamma}_{ab} U_{ibdi} + \delta_{ad} \underset{N}{\gamma}_{rs} U_{krsk} + \tfrac{1}{2} \underset{N}{\gamma}_{ad} U_{krrk} + D_{iadimn} - \tfrac{1}{2}\delta_{ab} D_{krrkmm} + O(k^{N+1} r^{-2}) \tag{A7}$$

Now, if the following relationships between the symbols U and D, and the symbols L and M defined in (38) and (47), are taken into account,

$$L_{abcd} \equiv U_{abcd} \tag{A8}$$

$$M_{abcd} \equiv D_{abcdmn} \tag{A9}$$

$$L^*_{bc} := L_{pbcp} \equiv U_{pbcp} \tag{A10}$$

$$M^*_{bc} := M_{pbcp} \equiv D_{pbcpmm} \tag{A11}$$

we obtain from (A7)

$$\begin{aligned} {}^{(\infty)}\underset{N}{\hat{G}}{}^{ab} = &- \underset{N}{\gamma}_{ij} L_{iabj} - \underset{N}{\gamma}_{bc} L^*_{ac} - \underset{N}{\gamma}_{bc} L^*_{ac} - \underset{N}{\gamma}_{ac} L^*_{bc} + \delta_{ab} \underset{N}{\gamma}_{rs} L^*_{rs} \\ &+ \tfrac{1}{2} \underset{N}{\gamma}_{ab} L^*_{rr} + M_{ab} + O(k^{N+1} r^{-2}) \end{aligned} \tag{A12}$$

This expression is always valid if the conditions established in Section 3.3 are satisfied.

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